Submodular Maximization with Matroid and Packing Constraints in Parallel
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Submodular Functions and Submodular Maximization

Submodular functions are functions that satisfy the diminishing returns property. Formally, they are defined as $F : 2^M \to \mathbb{R}$, such that $F(A \cup B) = F(A) + F(B) \leq F(A) + F(B)$, for all $A, B \subseteq [n]$. Equivalently, they satisfy the diminishing returns property: $F(A \cup \{e\}) - F(A) \geq F(B \cup \{e\}) - F(B)$ for all $A \subseteq B \subseteq [n], e \in [n]$.

Examples:

- Monotone f: continuous greedy
  $x = \arg \max \{\nabla f(x) \cdot v\}$
  - Frank-Wolfe constrained to nonnegative directions.
  - Makes at least as much progress as it would make by moving towards $x \lor x' \in P$, so can always gain $f(x) - f(x') - f(x') - f(x)$ per unit of step due to concavity.
  - Gain given by solving $\nabla f(x) = v$, then $v = \nabla f(x)'(1 - 1/e)$, so $v = (1 - 1/e)$.

- Non-monotone f: ramp step
  $x = (1 - x) \cdot \arg \max \{\nabla f(x) \cdot v\}$
  - Issue: since f is non-monotone, we do not have $f(x) \geq f(x')$, so cannot gain the same amount of progress.
  - Instead, use $f(x) \geq f(1 - ||x||_{\infty})$.
  - Ramp step, slows the growth of $||x||_{\infty}$.

- Can these algorithms be made practical?

Our Results

- We are concerned with maximizing $f$ in few parallel rounds of adaptivity (as defined by [1]). Pay for queries to $f$ and $\nabla f$.
- We only need to consider the more general setup for arbitrary functions concave along non-negative directions (DR-submodular).
- For monotone and non-monotone DR-submodular $f$, we obtain approximations that match the classical bounds up to additive $\epsilon$.

- Monotone $F$: 2-step algorithm gives $1/2$-approximation $|S|$ $\leq |S^*| - \epsilon$.
- Non-monotone $F$: $O\left(\frac{\log^3 n}{\epsilon^3}\right)$ time.
- New base $S \subseteq S^*$ can be reached from $S$.

Matroid Constraints

Observation 1: Greedy algorithm gives $1/2$-approximation [4].
$S' = S \cup \arg \max_{e \in \epsilon P} f(S \cup \{e\}) = f(S)$.

Proof idea: Maintain $S \in P$ and matroid base $O = S^* \cup S \setminus T$, where $T \subseteq S^* \setminus S$ is obtained via exchange property. Greedy enforces $f(S) \geq f(S^*) - f(O)$.

Observation 2: Running greedy $O(1/\epsilon)$ times improves to $1 - 1/\epsilon - \epsilon$.
Proof idea: Break greedy over $O(1/\epsilon)$ iterations.

Takeaway notes

- Continuous optimization makes discrete problems easy – the only property of $f$ we use is concavity along non-negative directions.
- Submodular maximization with packing constraints is no harder than solving positive LP’s in parallel.
- Extend techniques to other discrete problems? What else is there?

References