Improved Algorithms for Combinatorial Discrepancy

Adrian Vladu
adrian.vladu@irif.fr
IRIF, Université de Paris

Context

Discrepancy theory is a subfield of combinatorics which has branched in Computer Science due to several connections it has to geometric problems, randomized algorithms and complexity theory [13, 8].

A landmark result in the field is Spencer's celebrated *"Six standard deviations suffice"* [17]. In its simplest form, Spencer's paper considers a set system $\mathcal{S}$ of cardinality n over a ground set of n elements. The problem is to color each element of the ground set in red or blue, such that all the sets in the system are *balanced* in the sense that no set contains many more red than blue elements or vice-versa. The maximum imbalance of a set induced by this coloring is called the *discrepancy* of the coloring.

Spencer's result offers an important insight into the limitations of the tools we generally employ to prove the existence of mathematical objects. A standard method to show that there always exists a low discrepancy coloring is to prove that a random coloring produces one with nonzero probability. To this end, one shows that for each set in $\mathcal{S}$, a random coloring will have small discrepancy, with high probability. Applying a union bound turns this into a guarantee for all sets in $\mathcal{S}$. This standard approach shows that one can always produce a coloring which achieves discrepancy $O\left(\sqrt{n \log n}\right)$. [17] shows that this approach misses even better colorings. Using a difficult nonconstructive argument Spencer proves that in fact colorings with discrepancy $6\sqrt{n}$ exist. This result is tight up to constant factors, and exhibits an example where correlations between different sets in $\mathcal{S}$ can be exploited in order to overcome the limitations of the union bound technique, which essentially tries to ignore them.

Although in [17] it is conjectured that there are inherent limitations to finding a low discrepancy coloring, and no constructive proof exists, Bansal [2] proved the contrary by exhibiting a polynomial time algorithm whose output matches Spencer's bound, up to constant multiplicative factors. This provided a new direction for attacking open problems in the field, and was followed by a deluge of new algorithmic results [12, 9, 16, 11, 5, 3, 4].

This new algorithmic machinery presents potential for attacking outstanding open problems in the field. Of particular interest are the Beck-Fiala and Komlos conjecture, both of which are strengthenings of Spencer's theorem.

Goals

We aim to further extend algorithmic ideas involving convex geometry and optimization in order to obtain improved bounds and constructive methods for discrepancy problems. A lot of recent work [7, 6, 18, 1] was dedicated to solving the slightly easier random instances or online versions. We believe that certain algorithmic ideas, including a series of new constructive methods, have potential to yield immediate simplifications and improvements for these results. In the longer run, these may provide new techniques for tackling long standing open problems in the field.

The goal of this internship will be to study the relevant recent literature [16, 11, 3, 10, 7, 6, 18, 1, 14, 15], simplify existing techniques, and study certain particular instances of these problems that are believed to be hard instances. A successful internship would result in a technical writeup publishable in a quality theory conference.

Additional Information

We seek motivated candidates with a strong mathematical background. They should be familiar with optimization, probability and high dimensional geometry. In case of a successful project, the candidate will be encouraged to transition to a PhD. Students interested in this topic are encouraged to send an email to adrian.vladu@irif.fr.

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